# GAUSSIAN FINITE ADDITIVE AND MULTIPLICATIVE FREE DECONVOLUTION

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**ABSTRACT**

Random matrix and free probability theory have many fruitful applications in many research areas, such as digital communication, mathematical finance and nuclear physics. In particular, the concept of free deconvolution can be used to obtain the eigenvalue distributions of involved functionals of random matrices. Historically, free deconvolution has been applied in the asymptotic setting, i.e., when the size of the matrices tends to infinity. However, the validity of the asymptotic assumption is rarely met in practice. In this paper, we analyze the additive and multiplicative free deconvolution in the finite regime case when the involved matrices are Gaussian. In particular, we propose algorithmic methods to compute finite free deconvolution. The two methods are based on the moments method and the use of zonal polynomials.

## I. INTRODUCTION

The aim of this paper is to study additive and multiplicative free deconvolution in the finite regime case. The general idea of deconvolution relates to the following problem ([5]):
Given $\mathbf{A}$, $\mathbf{B}$ two $m \times m$ independent square hermitian (or symmetric) random matrices:
1) Can one derive the eigenvalue distribution of $\mathbf{A}$ from the ones of $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}$. If feasible, this operation is named additive free deconvolution,
2) Can one derive the eigenvalue distribution of $\mathbf{A}$ from the ones of $\mathbf{AB}$ and $\mathbf{B}$. If feasible, this operation is named multiplicative free deconvolution.
In this paper, we discuss the work which will be conducted during the thesis. In particular, we propose algorithmic methods to compute these operations for finite size matrices and discuss their limitations. Recently ([10], [12]), a unified framework for free deconvolution was proposed when $m \to \infty$ and for some particular cases of matrices $\mathbf{A}$ and $\mathbf{B}$ (for example, when $\mathbf{A}$ and $\mathbf{B}$ are free, or $\mathbf{A}$ is a random Vandermonde matrix and $\mathbf{B}$ is diagonal). In the asymptotic setting, the methods generally used to compute free convolution/deconvolution are the moments method and the Stieltjes transform method.

The moments method ([5]) gives relations between the moments of the different matrices involved. For a given $m \times m$ matrix $\mathbf{A}$, the $p$-th moment of $\mathbf{A}$ is defined as:

$$t_{\mathbf{A}}^{m,p} = \frac{1}{m} E\left[\mathrm{trace}(\mathbf{A}^p)\right] = \int \lambda^p d\rho(\lambda)$$

where $d\rho$ is the associated empirical mean measure defined as: $d\rho = E\left(\frac{1}{m}\sum_{i=1}^{m}\delta(\lambda-\lambda_i)\right)$ ($\lambda_i$ are the eigenvalues of $\mathbf{A}$. Quite remarkably, when $m \to \infty$, $t_{\mathbf{A}}^{m,p}$ converges almost surely to an analytical expression $t_{\mathbf{A}}^{p}$ and depends only on some specific parameters (maily the distribution of the entries of $\mathbf{A}$) of $\mathbf{A}$[1]. This enables to reduce the dimensionality of the problem and simplifies the computation of convolution of measures. Note that this method is analogue to computing moments of additive (or multiplicative) measures in the scalar case.
The Stieltjes transform method ([12]) provides an analytical transform in which probability measures of the eigenvalues of matrices are easy to compute. The Stieltjes transform of a probability measure $\mu$ is defined as

$$s_\mu(z) = \int \frac{1}{\lambda - z} dF^\mu(\lambda),$$

where $F^\mu$ is the cumulative distribution function of $\mu$. A simple inversion formula for the Stieltjes transform exists and is given by

$$f^\mu(\lambda) = \lim_{\omega \to =0^+} \frac{1}{\pi} Im[s_\mu(\lambda + j\omega)].$$

Here again, in many cases, one can compute an explicit form of the probability measure $\mu$ associated to the eigenvalues of $\mathbf{A}$ which depends only on the distribution of the entries of $\mathbf{A}$. The Stieltjes transform is analogue to the Fourier (Laplace) transform in the scalar case for computing the distribution of the sum (or product) of independent variables.
The goal of this paper is to generalize this framework in the case when $m$ is finite. As the problem is quite involved, we focus in the particular case of Gaussian matrices.

[1]Note that in the following, when speaking of moments of matrices, we refer to the moments of the associated measure.

## II. MODELS

The models that we will consider in this paper are of two types:

***Model** 1:* The correlated noise model:

$$\mathbf{Y} = \mathbf{R}^{\frac{1}{2}}\mathbf{S} \tag{1}$$

where the columns of the $m \times n$ matrix $\mathbf{S}$ are zero mean, independent complex Gaussian vectors with covariance matrix $\mathbf{I}$ and $\mathbf{R}$ is a $m \times m$ (deterministic) correlation matrix.
In this model, we will be interested in the Gram matrix $\mathbf{YY}^H$ associated to $\mathbf{Y}$, given by

$$\mathbf{YY}^H = \mathbf{R}^{\frac{1}{2}}\mathbf{SS}^H\mathbf{R}^{\frac{1}{2}}.$$

Multiplicative free deconvolution intends to express the joint eigenvalue distribution of $\mathbf{R}$ based only on the joint eigenvalue distribution of $\mathbf{YY}^H$.

***Model** 2:* The information plus noise model ([14]):

$$\mathbf{Y} = \mathbf{M} + \mathbf{R}^{\frac{1}{2}}\mathbf{S} \tag{2}$$

where $\mathbf{M}$ is a deterministic $m \times n$ matrix and the columns of the $m \times n$ matrix $\mathbf{S}$ are zero mean, independent complex Gaussian vectors with covariance matrix $\mathbf{I}$.
In this model, we will be interested in the Gram matrix $\mathbf{YY}^H$ associated to $\mathbf{Y}$, defined as

$$\mathbf{YY}^H = (\mathbf{M} + \mathbf{R}^{\frac{1}{2}}\mathbf{S})(\mathbf{M} + \mathbf{R}^{\frac{1}{2}}\mathbf{S})^H.$$

The aim of additive free deconvolution is to express the joint eigenvalue distribution of $\mathbf{MM}^H$ based only on the joint eigenvalue distribution of $\mathbf{YY}^H$ and $\mathbf{R}$.

## III. ASYMPTOTIC CASE

In this section, we shall review some classical results of free probability theory. The known literature about free convolution/deconvolution deals with the case of large random matrices when the dimensions go to infinity.
In the asymptotic setting, let us recall the results due to Voiculescu ([15]): for $\mathbf{A}_n, \mathbf{B}_n$ independent large $n \times n$ hermitian (or symmetric) random matrices (both of them having iid entries, or one of them having a distribution which is invariant under conjugation by any orthogonal matrix), if the eigenvalue distributions of $\mathbf{A}_n, \mathbf{B}_n$ converge, as $n$ tends to infinity, to some probability measures $\mu, \nu$, then the eigenvalue distribution of $\mathbf{A}_n + \mathbf{B}_n$ ($\mathbf{A}_n\mathbf{B}_n$) converges to a probability measure which depends only on $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$, which is denoted by $\mu_{\mathbf{A}} \boxplus \mu_{\mathbf{B}}$ ($\mu_{\mathbf{A}} \boxtimes \mu_{\mathbf{B}}$) and called the additive (multiplicative) free convolution of $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$. The idea of having one of the matrices which is unitarily invariant permits to have an eigenvector structure which is "deconnected" between matrices. Therefore, the knowledge of the eigen-structure of the involved matrices has no impact on the final result.

### III-A. Asymptotic additive free deconvolution: Definition

Practically, and without going to the strict definition provided in [15], the idea of additive free convolution stems from the fact that:

$$\begin{aligned} t^p_{\mathbf{A}+\mathbf{B}} &= \lim_{m\to\infty} \frac{1}{m}\mathrm{trace}\left((\mathbf{A}+\mathbf{B})^p\right) = \\ &= f(t^{(1)}_{\mathbf{A}}, \dots, t^{(p)}_{\mathbf{A}}, t^{(1)}_{\mathbf{B}}, \dots, t^{(p)}_{\mathbf{B}}) \end{aligned}$$

which means that we can express the moments of $\mathbf{A} + \mathbf{B}$ as a function of the moments of $\mathbf{A}$ and the moments of $\mathbf{B}$. Hence, when this happens, one is able by recursion to express all the moments of $\mathbf{A}$ with respect only to the moments of $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}$. Since the distribution of $\mathbf{A}+\mathbf{B}$ depends only on the probability measure associated with the moments of $\mathbf{A}$ and $\mathbf{B}$, one can define on the set of probability measures the following operation:
**Additive Free Convolution:** The additive free convolution of a measure $\mu$ and a measure $\nu$ is the measure $\rho$ such that $\rho = \mu \boxplus \nu$.
**Additive Free Deconvolution:** The additive free deconvolution of a measure $\rho$ by a measure $\nu$ is (when it exists) the only measure $\mu$ such that $\rho = \mu \boxminus \nu$.

### III-B. Asymptotic multiplicative free deconvolution: Definition

The multiplicative free convolution idea stems from the fact that:

$$\begin{aligned} t^p_{\mathbf{AB}} &= \lim_{m\to\infty} \frac{1}{m}\mathrm{trace}\left((\mathbf{AB})^p\right) = \\ &= f(t^{(1)}_{\mathbf{A}}, \dots, t^{(p)}_{\mathbf{A}}, t^{(1)}_{\mathbf{B}}, \dots, t^{(p)}_{\mathbf{B}}) \end{aligned}$$

which means that we can express the moments of $\mathbf{AB}$ as a function of the moments of $\mathbf{A}$ and the moments of $\mathbf{B}$. Once again, since the distribution of $\mathbf{AB}$ depends only on the probability measure associated with the moments of $\mathbf{A}$ and $\mathbf{B}$, one can define on the set of probability measures the following operation:
**Multiplicative Free Convolution** The multiplicative free convolution of a measure $\mu$ and a measure $\nu$ is the measure $\rho$ such that $\rho = \mu \boxtimes \nu$.
**Multiplicative Free Deconvolution** The multiplicative free deconvolution of a measure $\rho$ and a measure $\nu$ is the measure $\mu$ such that $\rho = \mu \boxslash \nu$.

In the next two sections, we will recall the main results and the algorithmic methods used in the asymptotic case to compute additive and multiplicative free deconvolution.

### III-C. Moments method

In this section, let us describe the moments method.

***Additive Case***

The moments method is based on the relation between the moments $t^p_{\mathbf{A}}$ and the free cumulants $d^p_{\mathbf{A}}$ of a matrix $\mathbf{A}$. They can be deduced one from each other as their power series, that we denote by $T_{\mathbf{A}}(z) = \sum_{p\geq 1} t^p_{\mathbf{A}} z^p$ and $D_{\mathbf{A}}(z) = \sum_{p\geq 1} d^p_{\mathbf{A}} z^p$ are linked by the following relation

$$D_{\mathbf{A}}(z(T_{\mathbf{A}}(z)) + 1)) = T_{\mathbf{A}}(z).$$

Hence, we have the relations for all $p \geq 0$

$$t^0_{\mathbf{A}} = 1$$

$$t^p_{\mathbf{A}} = d^p_{\mathbf{A}} + \sum_{k=1}^{n-1} \left[ \sum_{\substack{p_1, \dots, p_k \geq 0 \\ p_1 + \dots + p_k = p}} t^{p_1}_{\mathbf{A}} \dots t^{p_k}_{\mathbf{A}} \right].$$

The following characterization enables to compute easily the additive free convolution using free cumulants.

***Theorem 1:*** Given $\mathbf{A}$ and $\mathbf{B}$ free random matrices, $\mu_{\mathbf{A}\boxplus\mathbf{B}}$ is the only law such that for all $p \geq 1$

$$d^p_{\mathbf{A}\boxplus\mathbf{B}} = d^p_{\mathbf{A}} + d^p_{\mathbf{B}} \tag{3}$$

Hence, the deconvolution $\mu_{\mathbf{A}+\mathbf{B}\boxminus\mathbf{B}}$ of $\mu_{\mathbf{A}+\mathbf{B}}$ by $\mu_{\mathbf{B}}$ is characterized by the fact that for all $p \geq 1$

$$d^p_{(\mathbf{A}+\mathbf{B})\boxminus\mathbf{B}} = d^p_{\mathbf{A}+\mathbf{B}} - d^p_{\mathbf{B}}. \tag{4}$$

The implementation of additive free deconvolution is based on the following steps: for the two matrices $(\mathbf{A}+\mathbf{B})$ and $\mathbf{B}$, we first compute the cumulants. Considering the relation between the cumulants and the moments, we can obtain information about the distribution of the eigenvalues of $\mathbf{A}$.

***Multiplicative Case***

The moments method, in the multiplicative case, is based on the relation between the moments $t^p_{\mathbf{A}}$ and the coefficients $l^p_{\mathbf{A}}$ of the $S$-transform of measure associated to $\mathbf{A}$. They can be deduced one from each other as their power series, that we denote by $T_{\mathbf{A}}(z) = \sum_{p\geq 1} t^p_{\mathbf{A}} z^p$ and $S_{\mathbf{A}}(z) = \sum_{p\geq 1} l^p_{\mathbf{A}} z^{p-1}$, are linked by the following relation

$$T_{\mathbf{A}}(z) S_{\mathbf{A}}(T_{\mathbf{A}}(z)) = z(1 + T_{\mathbf{A}}(z))).$$

Equivalently, we have the relations for all $p \geq 1$

$$t^1_{\mathbf{A}} l^1_{\mathbf{A}} = 1$$

$$l^p_{\mathbf{A}} = \sum_{k=1}^{p+1} l^k_{\mathbf{A}} + \left[ \sum_{\substack{p_1, \dots, p_k \geq 1 \\ p_1 + \dots + p_k = p+1}} t^{p_1}_{\mathbf{A}} \dots t^{p_k}_{\mathbf{A}} \right].$$

Hence, we can compute multiplicative free convolution by the following characterization.

***Theorem 2:*** Given $\mathbf{A}$ and $\mathbf{B}$ free random matrices, $\mathbf{A}\boxtimes\mathbf{B}$ is the only law such that:

$$S_{\mathbf{A}\boxtimes\mathbf{B}} = S_{\mathbf{A}} S_{\mathbf{B}}$$

The multiplicative free deconvolution $\mu_{\mathbf{AB}\boxslash\mathbf{B}}$ of $\mu_{\mathbf{AB}}$ by $\mu_{\mathbf{B}}$ is characterized by the fact that for all $p \geq 1$

$$l^p_{(\mathbf{AB})\boxslash\mathbf{B}} l^1_{\mathbf{B}} = l^p_{\mathbf{AB}} - \sum_{k=1}^{p-1} l^k_{(\mathbf{AB})\boxslash\mathbf{B}} l^{p+1-k}_{\mathbf{B}}. \tag{5}$$

### III-D. Stieltjes transform method

The aim of this section is to present the Stieltjes transform method in the additive and in the multiplicative case to compute free convolution/deconvolution.

***Additive case: $R$-transform*** Let $\rho$ be a probability measure. The $R$-transform, denoted by $R_\rho$, is defined by

$$G_\rho\left(\frac{R_\rho(z)+1}{z}\right) = z,$$

where $G_\rho = \int_{t\in\mathbb{R}} \frac{1}{z-t} d\rho(t)$ is the Cauchy transform. The additive free convolution of the measures $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$ can be computed through the $R$-transform by the following property:

$$R_{\mathbf{A}\boxplus\mathbf{B}} = R_{\mathbf{A}} + R_{\mathbf{B}}.$$

In [6], it is proved how for any probability measure we can recover $G$ from $R_{\mathbf{A}\boxplus\mathbf{B}}$. Hence, in our case, we can compute the additive free deconvolution by recovering $R_{\mathbf{A}}$ from $R_{\mathbf{A}\boxplus\mathbf{B}}$ and $R_{\mathbf{B}}$. Unfortunately, from an algorithmic point of view, one has to solve a polynomial equation, which does not always have an explicit solution.

***Multiplicative case: $S$-transform***

The analytical method for the computation of free multiplicative convolution/deconvolution is the $S$-transform. The $S$-transform $S$ of a probability measure $\rho$ ($\neq \delta_0$) is defined in $\mathbb{C} \setminus [0, \infty)$ by

$$M\left(\frac{z}{1+z} S(z)\right) = z,$$

where $M(z) = \int_{t\in\mathbb{R}} \frac{zt}{1-zt} d\rho(t)$. The importance of the $S$-transform comes from the following multiplicative property:

$$S_{\mathbf{A}\boxtimes\mathbf{B}} = S_{\mathbf{A}} S_{\mathbf{A}},$$

which can be applied to compute free multiplicative convolution and deconvolution. Here again, let us recall that this method works only in some limited cases, because the $S$-transforms are almost never explicit.

## IV. FINITE CASE

Our goal is to propose a pratical approach to compute free deconvolution in the finite case. In the same vein, we propose two approaches for computing the eigenvalue distribution. The first is based on the invariant polynomials and the computation of the zonal polynomials. The second one is based on the moments method, whose difficulty relies on the calculation of partitions.

### IV-A. Definitions

Before introducing the tools for finite deconvolution, let us first recall some definitions in the finite regime case.

***Definition 1:*** Define the $m \times n$ matrix $\mathbf{Y}$ as in (1), the $m \times m$ random matrix $\mathbf{W} = \mathbf{Y}\mathbf{Y}^H$ is a central complex Wishart matrix with $n$ degrees of freedom and covariance matrix $\mathbf{R}$, denoted by ($\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{R})$).

***Definition 2:*** Define the $m \times m$ matrix $\mathbf{Y}$ as in (2), the $m \times m$ random matrix $\mathbf{W} = \mathbf{Y}\mathbf{Y}^H$ is a noncentral complex Wishart matrix with $n$ degrees of freedom and noncentrality matrix $\mathbf{\Omega} = \mathbf{R}^{-1}\mathbf{M}^H\mathbf{M}$, ($\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{\Omega}, \mathbf{R})$).

### IV-B. Zonal Polynomials

In this section, we look at the operation of convolution through the zonal polynomials. The probability distributions of complex random matrices are often expressed in terms of complex hypergeometric functions of matrix arguments, which are defined by their expansions in zonal polynomials ([8]). The complex zonal polynomials ([9]) are multivariate polynomials of the eigenvalues of a hermitian matrix and their definition arise from group representation theory ([8]).
Let $\kappa = (k_1, \ldots, k_m)$ be a partition of the integer $k$ such as $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$ and $k_1 + \cdots + k_m = m$.
The complex zonal polynomial[2] of a complex matrix $\mathbf{X} \in \mathbb{C}^{m \times m}$ is defined [8] as

$$C_\kappa(\mathbf{X}) = \chi_{[\kappa]}(1)\chi_{[\kappa]}(\mathbf{X}) \tag{6}$$

where $\chi_{[\kappa]}$ is the dimension of the representation $[\kappa]$ of the symmetric group on $k$ symbols given by

$$\chi_{[\kappa]}(1) = k! \frac{\prod_{i<j}^m (k_i - k_j - i + j)}{\prod_{i=1}^m (k_i + m - i)!} \tag{7}$$

and $\chi_{[\kappa]}(\mathbf{X})$ is the character of the representation $[\kappa]$ of the linear group given as a symmetric function of the eigenvalues $\mu_1, \ldots, \mu_m$ of $\mathbf{X}$ by

$$\chi_{[\kappa]}(\mathbf{X}) = \frac{\det[\mu_i^{k_j+m-j}]}{\det[\mu_i^{m-j}]} \tag{8}$$

[2]In general, the complex and the real zonal polynomial are denoted, respectively, by $\tilde{C}_\kappa(\mathbf{X})$ and $C_\kappa(\mathbf{X})$. In this paper we use the notation $C_\kappa(\mathbf{X})$ for the complex zonal polynomials because we are not considering the real case.

In the finite case, free deconvoltion can be computed using zonal polynomials. In fact, the eigenvalues densities of complex noncentral Wishart matrices can be expressed by zonal polynomials, introduced by Davis in ([1], [2]). These polynomials are symmetric in the eigenvalues of a complex matrix and have two matrix arguments, which extend the single matrix argument of zonal polynomial, i.e., we denote by $C_\phi^{\kappa,\tau}(\mathbf{X}, \mathbf{Y})$ homogeneous polynomials of degree $k$ and $t$ in the elements of the $m \times m$ symmetric complex matrices $\mathbf{X}$ and $\mathbf{Y}$, where $\kappa$, $\tau$ and $\phi$ are ordered partitions of the non negative integers $k$, $t$ and $f = k + t$, respectively, into not more than $m$ parts. These polynomials are called invariant beacuse they are invariant under the simultaneous transformations

$$\mathbf{X} \to \mathbf{E}^H\mathbf{X}\mathbf{E},\ \mathbf{Y} \to \mathbf{E}^H\mathbf{Y}\mathbf{E},\ \mathbf{E} \in U(m).$$

In the following theorem ([9]), the joint density of the eigenvalues of a complex noncentral Wishart matrix is given in terms of zonal polynomials.

***Theorem 3:*** Let $n \geq m$ be and consider the $m \times m$ positive definite Hermitian matrix $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{\Omega}, \mathbf{R})$. Then the joint density of the eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ of $\mathbf{W}$ is

$$f(\mathbf{\Lambda}) = \frac{\pi^{m(m-1)}|\mathbf{R}|^{-n}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)} e^{tr(-\mathbf{\Omega})} \prod_{k=1}^m {\lambda_k}^{n-m} \prod_{k<l}^m \lambda_k - {\lambda_l}^2 \times \\ \times \sum_{k,t=0}^\infty \sum_{\kappa,\tau;\phi\in\kappa,\tau} \frac{C_\phi^{\kappa,\tau}(-\mathbf{R}^{-1}, \mathbf{\Omega}\mathbf{R}^{-1}) C_\phi^{\kappa,\tau}(\mathbf{\Lambda}, \mathbf{\Lambda})}{k!t![n]_\tau C_\phi(\mathbf{I}_m)} \tag{9}$$

where $[n]_\tau$ is the complex multivariate hypergeometric coefficient defined as $[n]_\tau = \prod_{i=1}^m (n - i + 1)_{t_i}$ with $(n)_t = n(n+1)\ldots(n+t-1)$ and $C_\kappa(\mathbf{I}_m)$ is given by

$$C_\phi(\mathbf{I}_m) = 2^{2f} f! \left(\frac{1}{2}m\right)_\phi \times \\ \times \frac{\prod_{i<j}^r (2f_i - 2f_j - i + j)}{\prod_{i=1}^r (2f_i + m - i)!} \tag{10}$$

with

$$\left(\frac{1}{2}m\right)_\phi = \prod_{i=1}^r \left(\frac{1}{2}(m - i + 1)\right)_{f_i} \tag{11}$$

where $r$ are the nonzero parts of $\phi$.

Hence, we observe from (9) that the operation of deconvolution consists in deriving information about the noncentrality matrix $\mathbf{\Omega}$ in function of the eigenvalues of the Wishart matrix considered. In order to do this, it is necessary to express the invariant polynomials. This is not a simple task ([4]) and can be related to the works of Davis ([2], [3]). There are some particular cases where this can be done, typically:

a) $\mathbf{R}=\mathbf{I}$. In this case, (9) is given by

$$f(\mathbf{\Lambda})=\frac{\pi^{m(m-1)}e^{tr(-\mathbf{M}^H\mathbf{M})}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\prod_{k=1}^{m}{\lambda_k}^{n-m}\prod_{k<l}^{m}\lambda_k-{\lambda_l}^2 \tag{12}$$

$$\times\sum_{k,t=0}^{\infty}\sum_{\kappa,\tau;\phi\in\kappa,\tau}\frac{C_{\phi}^{\kappa,\tau}(-\mathbf{I},\mathbf{M^HM})C_{\phi}^{\kappa,\tau}(\mathbf{\Lambda},\mathbf{\Lambda})}{k!t![n]_{\tau}C_{\phi}(\mathbf{I}_m)}$$

b) $\mathbf{M}=\mathbf{0}$: in this case, (9) is given by

$$f(\mathbf{\Lambda})=\frac{\pi^{m(m-1)}|\mathbf{R}|^{-n}}{\tilde{\Gamma}_m(n)\tilde{\Gamma}_m(m)}\prod_{k=1}^{m}{\lambda_k}^{n-m}\prod_{k<l}^{m}\lambda_k-{\lambda_l}^2 \tag{13}$$

$$\times\sum_{k,t=0}^{\infty}\sum_{\kappa,\tau;\phi\in\kappa,\tau}\frac{C_{\phi}^{\kappa,\tau}(-\mathbf{R}^{-1},\mathbf{0})C_{\phi}^{\kappa,\tau}(\mathbf{\Lambda},\mathbf{\Lambda})}{k!t![n]_{\tau}C_{\phi}(\mathbf{I}_m)}$$

The main difficulty of finite free deconvolution is related to the implementation of these functions.

### IV-C. Moments Method

In this section, we explain the moments method which we propose to compute free deconvolution in the finite case. For example, in the model (1), we need to compute the moments of $\mathbf{YY}^H$, given by

$$t_{\mathbf{Y}}^{m,n,p}=\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{YY}^H\right)^p\right)\right]$$

and relate these moments to those of $\mathbf{R}$ and $\mathbf{S}$. Note that this can be easilly done when $\mathbf{R}=\mathbf{I}$.
Suppose that $\mathbf{Y}$ is a $m\times n$ standard Gaussian matrix. We can consider $S_p$ the set of permutations of $p$ elements $\{1,2,\ldots,p\}$, ([11]). For $\pi\in S_p$, we denote by $\hat{\pi}$ the permutation in $S_{2p}$ defined by

$$\hat{\pi}(2j-1)=2\pi^{-1}(j),\ (j\in\{1,2,\ldots,p\}) \tag{14}$$
$$\hat{\pi}(2j)=2\pi(j)-1,\ (j\in\{1,2,\ldots,p\}). \tag{15}$$

Let $k(\hat{\pi})$ and $l(\hat{\pi})$ be the number of equivalence classes of $\sim_p$ consisting of even numbers and odd numbers, respectively. Then, one can show that:

$$\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{YY}^H\right)^p\right)\right]=\frac{1}{mn^p}\sum_{\pi\in S_p}n^{k(\hat{\pi})}m^{l(\hat{\pi})}. \tag{16}$$

For example, the first fourth moments are given by

$$\frac{1}{m}E\left[\mathrm{trace}\left(\frac{1}{n}\mathbf{YY}^H\right)\right]=1,$$

$$\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{YY}^H\right)^2\right)\right]=c+1,$$

$$\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{YY}^H\right)^3\right)\right]=c^2+3c+1+\frac{1}{n^2},$$

$$\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{YY}^H\right)^4\right)\right]=c^3+6c^2+6c+1+\frac{5(1+c)}{n^2},$$

where $c=\frac{m}{n}$.

Future work in this Phd thesis will focus on the calculation of

$$\frac{1}{m}E\left[\mathrm{trace}\left(\left(\frac{1}{n}\mathbf{R}^{\frac{1}{2}}\mathbf{SS}^H\mathbf{R}^{\frac{1}{2}}\right)^p\right)\right]$$

with respect to the eigenvalues of $\mathbf{R}$. This will also be extended to the information plus noise model.

## V. CONCLUSION

In this paper, we have proposed to extend free deconvolution to finite matrices. For this, we have proposed two algorithmic methods: the moments method and the invariant polynomials method. Each one of those methods present some difficulties with pratical implementation problems. Typically, in the case of a Wishart matrix, such result exists already. Although we focus, for tractability, on the Gaussian complex law, the results could be extended to the non complex case. Applications to the cognitive radio context will be detailed at the end of the Phd.

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